Remarks on some quasilinear equations with gradient terms and measure data

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, H a Caratheodory function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and μ a bounded Radon measure in Ω. We study the problem

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Δ_p is the p-Laplacian ($p > 1$), and we emphasize the case $H(x, u, \nabla u) = \pm |\nabla u|^q$ ($q > 0$). We obtain an existence result under subcritical growth assumptions on H, we give necessary conditions of existence in terms of capacity properties, and we prove removability results of eventual singularities. In the supercritical case, when $\mu \geqq 0$ and H is an absorption term, i.e. $H \geqq 0$, we give two sufficient conditions for existence of a nonnegative solution.

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N \geqq 2)$. In this article we consider problems of the form

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator, with $1 < p \leqq N$, H is a Caratheodory function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and μ is a possibly signed Radon measure on Ω. We study the existence of solutions for the Dirichlet problem in Ω

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and some questions of removability of the singularities. Our main motivation is the case where μ is nonnegative, H involves only ∇u, and either H is nonnegative, hence H is an absorption term, or H is nonpositive, hence H is a source one. The model cases are

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \tag{1.3}$$

where $q > 0$, for the absorption case and

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega. \tag{1.4}$$

for the source case.

The equations without gradient terms,

$$-\Delta_p u + H(x, u) = \mu \quad \text{in } \Omega, \tag{1.5}$$

such as the quasilinear Emden-Fowler equations

$$-\Delta_p u \pm |u|^{Q-1} u = \mu \quad \text{in } \Omega,$$

where $Q > 0$, have been the object of a huge literature when $p = 2$. In the general case $p > 1$, among many works we refer to [5], [6], [7] and the references therein, and to [8] for new recent results in the case of absorption.

We set

$$Q_c = \frac{N(p-1)}{N-p}, \qquad q_c = \frac{N(p-1)}{N-1}, \qquad (Q_c = \infty \text{ if } p = N), \qquad \tilde{q} = p - 1 + \frac{p}{N} \tag{1.6}$$

(hence $q_c, \tilde{q} < p < N$ or $q_c = \tilde{q} = p = N$), and

$$q_* = \frac{q}{q+1-p}, \tag{1.7}$$

(thus $q_* = q'$ in case $p = 2$).

In Section 2 we recall the main notions of solutions of the problem $-\Delta_p u = \mu$, such as weak solutions, renormalized or locally renormalized solutions, and convergence results. In Section 3 we prove a general existence result for problem (1.2) in the subcritical case, see Theorem 3.1. Then in Section 4 we give necessary conditions for existence and removability results for the local solutions of problem (1.1), extending former results of [20] and [39], see Theorem 4.5. In Section 5 we study the problem (1.2) in the supercritical case, where many questions are still open. We give two partial results of existence in Theorems 5.5 and 5.8. Finally in Section 5 we make some remarks of regularity for the problem

$$-\Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \Omega.$$

2 Notions of solutions

Let ω be any domain of $\mathbb{R}^N$. For any $r>1$, the capacity $cap_{1,r}$ associated to $W_0^{1,r}(\omega)$ is defined by

$$cap_{1,r}(K,\omega)=\inf\left\{\|\psi\|^r_{W_0^{1,r}(\omega)}:\psi\in\mathcal{D}(\omega),\chi_K\leq\psi\leq 1\right\},$$

for any compact set $K\subset\omega$, and then the notion is extended to any Borel set in ω. In $\mathbb{R}^N$ we denote by G_1 the Bessel kernel of order 1 (defined by $\widehat{G_1}(y)=(1+|y|^2)^{-1/2}$), and we consider the Bessel capacity defined for any compact $K\subset\mathbb{R}^N$ by

$$Cap_{1,r}(K,\mathbb{R}^N)=\inf\left\{\|f\|^r_{L^r(\mathbb{R}^N)}:f\geqq 0,G_1*f\geqq\chi_K\right\}.$$

On $\mathbb{R}^N$ the two capacities are equivalent, see [2].

We denote by $\mathcal{M}(\omega)$ the set of Radon measures in ω, and $\mathcal{M}_b(\omega)$ the subset of bounded measures, and define $\mathcal{M}^+(\omega)$, $\mathcal{M}_b^+(\omega)$ the corresponding cones of nonnegative measures. Any measure $\mu\in\mathcal{M}(\omega)$ admits a positive and a negative parts, denoted by μ^+ and μ^-. For any Borel set E, $\mu\llcorner E$ is the restriction of μ to E; we say that μ is concentrated on E if $\mu=\mu\llcorner E$.

For any $r>1$, we call $\mathcal{M}^r(\omega)$ the set of measures $\mu\in\mathcal{M}(\omega)$ which do not charge the sets of null capacity, that means $\mu(E)=0$ for every Borel set $E\subset\omega$ with $cap_{1,r}(E,\omega)=0$. Any measure concentrated on a set E with $cap_{1,r}(E,\omega)=0$ is called r-singular. Similarly we define the subsets $\mathcal{M}_b^r(\omega)$ and $\mathcal{M}_b^{r+}(\omega)$.

For fixed $r>1$, any measure $\mu\in\mathcal{M}(\omega)$ admits a unique decomposition of the form $\mu=\mu_0+\mu_s$, where $\mu_0\in\mathcal{M}^r(\omega)$, and $\mu_s=\mu_s^+-\mu_s^-$ is r-singular. If $\mu\geqq 0$, then $\mu_0\geqq 0$ and $\mu_s\geqq 0$.

Remark 2.1 *Any measure $\mu\in\mathcal{M}_b(\omega)$ belongs to $\mathcal{M}^r(\omega)$ if and only if there exist $f\in L^1(\omega)$ and $g\in(L^{r'}(\omega))^N$ such that $\mu=f+divg$, see [11, Theorem 2.1]. However this decomposition is not unique; if μ is nonnegative there exists a decomposition such that f is nonnegative, but one cannot ensure that $divg$ is nonnegative.*

For any $k>0$ and $s\in\mathbb{R}$, we define the truncation $T_k(s)=\max(-k,\min(k,s))$. If u is measurable and finite a.e. in ω, and $T_k(u)$ belongs to $W_0^{1,p}(\omega)$ for every $k>0$, one can define the gradient ∇u a.e. in ω by $\nabla T_k(u)=\nabla u.\chi_{\{|u|\leqq k\}}$ for any $k>0$.

For any $f\in\mathcal{M}^+\left(\mathbb{R}^N\right)$, we denote the Bessel potential of f by $J_1(f)=G_1*f$.

2.1 Renormalized solutions

Let $\mu\in\mathcal{M}_b(\Omega)$. Let us recall some known results for the problem

$$-\Delta_p u=\mu\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,\tag{2.1}$$

Under the assumption $p>2-1/N$, from [9], problem (2.1) admits a solution $u\in W_0^{1,r}(\Omega)$ for every $r\in[1,q_c)$, satisfying the equation in $\mathcal{D}'(\Omega)$. When $p<2-1/N$, then $q_c<1$; this leads to introduce the concept of renormalized solutions developed in [16], see also [33], [44]. Here we recall one of their definitions, among four equivalent ones given in [16].

Definition 2.2 *Let $\mu = \mu^0 + \mu_s \in \mathcal{M}_b(\Omega)$, where $\mu^0 \in \mathcal{M}^p(\Omega)$ and $\mu_s = \mu_s^+ - \mu_s^-$ is p-singular. A function u is a **renormalized solution,** called **R-solution** of problem (2.1), if u is measurable and finite a.e. in Ω, such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for any $k > 0$, and $|\nabla u|^{p-1} \in L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$; and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in W^{1,s}(\Omega)$ for some $s > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla(h(u)\varphi)dx = \int_\Omega h(u)\varphi d\mu_0 + h(\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^-. \tag{2.2}$$

As a consequence, any R-solution u of problem (2.1) satisfies $|u|^{p-1} \in L^\sigma(\Omega), \forall\sigma \in [1, N/(N-p)$. More precisely, u and $|\nabla u|$ belong to some Marcinkiewicz spaces

$$L^{s,\infty}(\Omega) = \left\{ u \text{ measurable in } \Omega : \sup_{k>0} k^s \left|\{x \in \Omega : |u(x)| > k\}\right| < \infty \right\},$$

see [9], [5], [16], [27], and one gets useful convergence properties, see [16, Theorem 4.1 and §5] for the proof:

Lemma 2.3 *(i) Let $\mu \in \mathcal{M}_b(\Omega)$ and u be any R-solution of problem (2.1). Then for any $k > 0$,*

$$\frac{1}{k}\int_{\{m \leqq u \leqq m+k\}} |\nabla u|^p \, dx \leq |\mu|(\Omega), \forall m \geqq 0.$$

If $p < N$, then $u \in L^{Q_c,\infty}(\Omega)$ and $|\nabla u| \in L^{q_c,\infty}(\Omega)$,

$$|\{|u| \geqq k\}| \leqq C(N,p)k^{-Q_c}(|\mu|(\Omega))^{\frac{N}{N-p}}, \qquad |\{|\nabla u| \geqq k\}| \leqq C(N,p)k^{-q_c}(|\mu|(\Omega))^{\frac{N}{N-1}}. \tag{2.3}$$

If $p = N$ (where u is unique), then for any $r > 1$ and $s \in (1, N)$,

$$|\{|u| \geqq k\}| \leqq C(N,p,r)k^{-r}(|\mu|(\Omega))^{\frac{r}{p-1}}, \qquad |\{|\nabla u| \geqq k\}| \leqq C(N,p,s)k^{-N}(|\mu|(\Omega))^{\frac{s}{N-1}}. \tag{2.4}$$

(ii) Let (μ_n) be a sequence of measures $\mu_n \in \mathcal{M}_b(\Omega)$, uniformly bounded in $\mathcal{M}_b(\Omega)$, and u_n be any R-solution of

$$-\Delta_p u_n = \mu_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Then there exists a subsequence (μ_ν) such that (u_ν) converges a.e. in Ω to a function u, such that $T_k(u) \in W_0^{1,p}(\Omega)$, and $(T_k(u_\nu))$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_k(u)$, and (∇u_ν) converges a.e. in Ω to ∇u.

Remark 2.4 *These properties do not require any regularity of Ω. If $\mathbb{R}^N \backslash \Omega$ is geometrically dense, i.e. there exists $c > 0$ such that $|B(x,r)\backslash\Omega| \geqq cr^N$ for any $x \in \mathbb{R}^N\backslash\Omega$ and $r > 0$, then (2.4) holds with $s = N$, and C depends also on the geometry of Ω. Then $|\nabla u| \in L^{N,\infty}(\Omega)$, hence $u \in BMO(\Omega)$, see [17], [27].*

Next we recall the fundamental stability result of [16, Theorem 3.1]:

Definition 2.5 *For any measure $\mu = \mu^0 + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega)$, where $\mu^0 = f - divg \in \mathcal{M}^p(\Omega)$, and μ_s^+, μ_s^- are p-singular we say that a sequence (μ_n) is a **good approximation** of μ in $\mathcal{M}_b(\Omega)$ if it can be decomposed as*

$$\mu_n = \mu_n^0 + \lambda_n - \eta_n, \quad \textit{with} \quad \mu_n^0 = f_n - divg_n, \quad f_n \in L^1(\Omega), \quad g_n \in (L^{p'}(\Omega))^N, \quad \lambda_n, \eta_n \in \mathcal{M}_b^+(\Omega), \tag{2.5}$$

such that (f_n) converges to f weakly in $L^1(\Omega)$, (g_n) converges to g strongly in $(L^{p'}(\Omega))^N$ and $(divg_n)$ is bounded in $\mathcal{M}_b(\Omega)$, and (ρ_n) converges to μ_s^+ and (η_n) converges to μ_s^- in the narrow topology.

Theorem 2.6 ([16]) *Let $\mu \in \mathcal{M}_b(\Omega)$, and let (μ_n) be a good approximation of μ. Let u_n be a R-solution of*

$$-\Delta_p u_n = \mu_n \quad \textit{in } \Omega, \qquad u_n = 0 \quad \textit{on } \partial\Omega.$$

Then there exists a subsequence (u_ν) converging a.e. in Ω to a R-solution u of problem (2.1). And $(T_k(u_\nu))$ converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$.

Remark 2.7 *As a consequence, for any measure $\mu \in \mathcal{M}_b(\Omega)$, there exists at least a solution of problem (2.1). Indeed, it is pointed out in [16] that any measure $\mu \in \mathcal{M}_b(\Omega)$ can be approximated by such a sequence: extending μ by 0 to $\mathbb{R}^N$, one can take $g_n = g$, $f_n = \rho_n * f$, $\lambda_n = \rho_n * \mu_s^+$, $\eta_n = \rho_n * \mu_s^-$, where (ρ_n) is a regularizing sequence; then $f_n, \lambda_n, \eta_n \in C_b^\infty(\Omega)$. Notice that this approximation does not respect the sign: $\mu \in \mathcal{M}_b^+(\Omega)$ does not imply that $\mu_n \in \mathcal{M}_b^+(\Omega)$.*

In the sequel we precise the approximation property, still partially used in [19, Theorem 2.18] for problem (1.5).

Lemma 2.8 *Let $\mu \in \mathcal{M}_b(\Omega)$. Then*

(i) there exists a sequence (μ_n) of good approximations of μ, such $\mu_n \in W^{-1,p'}(\Omega)$, and μ_n^0 has a compact support in Ω, $\lambda_n, \eta_n \in C_b^\infty(\Omega)$, (f_n) converges to f strongly in $L^1(\Omega)$, and

$$|\mu_n|(\Omega) \leqq 4\,|\mu|(\Omega), \quad \forall n \in \mathbb{N} \tag{2.6}$$

Moreover, if $\mu \in \mathcal{M}_b^+(\Omega)$, then one can find the approximation such that $\mu_n \in \mathcal{M}_b^+(\Omega)$ and (μ_n) is nondecreasing.

(ii) there exists another sequence (μ_n) of good approximations of μ, with , with $f_n, g_n \in \mathcal{D}(\Omega)$, $\lambda_n, \eta_n \in C_b^\infty(\Omega)$, such that (f_n) converges to f strongly in $L^1(\Omega)$, satisfying (2.6); if $\mu \in \mathcal{M}_b^+(\Omega)$, one can take $\mu_n^0 \in \mathcal{D}^+(\Omega)$.

Proof. (i) Let $\mu = \mu^0 + \mu_s^+ - \mu_s^-$, where $\mu^0 \in \mathcal{M}^p(\Omega)$, μ_s^+, μ_s^- are p-singular and $\mu_1 = (\mu^0)^+, \mu_2 = (\mu^0)^-$; thus $\mu_1(\Omega) + \mu_2(\Omega) + \mu_s^+(\Omega) + \mu_s^-(\Omega) \leqq 2\,|\mu(\Omega)|$. Following [11], for $i = 1, 2$, one has

$$\mu_i = \varphi_i \gamma_i, \qquad \text{with } \gamma_i \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega) \text{ and } \varphi_i \in L^1(\Omega, \gamma_i).$$

Let $(K_n)_{n \geqq 1}$ be an increasing sequence of compacts of union Ω; set

$$\nu_{1,i} = T_1(\varphi_i \chi_{K_1})\gamma_i, \quad \nu_{n,i} = T_n(\varphi_i \chi_{K_n})\gamma_i - T_{n-1}(\varphi_i \chi_{K_{n-1}})\gamma_i, \quad \mu_{n,i}^0 = \sum_1^n \nu_{n,i} = T_n(\varphi_i \chi_{K_n})\gamma_i.$$

Thus $\mu_{n,i}^0 \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$. Regularizing by (ρ_n), there exists $\phi_{n,i} \in \mathcal{D}^+(\Omega)$ such that $\|\phi_{n,i} - \nu_{n,i}\|_{W^{-1,p'}(\Omega)} \leqq 2^{-n}\mu_i(\Omega)$. Then $\xi_{n,i} = \sum_1^n \phi_{k,i} \in \mathcal{D}^+(\Omega)$; $(\eta_{n,i})$ converges strongly in $L^1(\Omega)$ to a function ξ_i and $\|\xi_{n,i}\|_{L^1(\Omega)} \leqq \mu_i(\Omega)$. Also setting

$$G_{n,i} = \mu_{n,i}^0 - \xi_{n,i} = \sum_1^n (\nu_{n,i} - \phi_{k,i}) \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega),$$

then $(G_{n,i})$ converges strongly in $W^{-1,p'}(\Omega)$ to some G_i, and $\mu_i = \xi_i + G_i$, and $\|G_{n,i}\|_{\mathcal{M}_b(\Omega)} \leqq 2\mu_i(\Omega)$. Otherwise $\lambda_n = \rho_n * \mu_s^+$ and $\eta_n = \rho_n * \mu_s^- \in C_b^\infty(\Omega)$ converge respectively to μ_s^+, μ_s^- in the narrow topology, with $\|\lambda_n\|_{L^1(\Omega)} \leqq \mu_s^+(\Omega)$, $\|\eta_n\|_{L^1(\Omega)} \leqq \mu_s^-(\Omega)$. Then we set

$$\mu_n = \mu_n^0 + \rho_n - \eta_n \qquad \text{with } \mu_n^0 = \xi_n + G_n,\ \xi_n = \xi_{n,1} - \xi_{n,2} \in \mathcal{D}(\Omega), \quad G_n = G_{n,1} - G_{n,2} \in W^{-1,p'}(\Omega)$$

thus μ_n^0 has a compact support. Moreover $\mu_0 = \xi + G$ with $\xi = \xi_1 - \xi_2 \in \mathcal{D}(\Omega)$, and $G = G_1 - G_2 = \varphi + divg$ for some $\varphi \in L^{p'}(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, and (G_n) converges to G in $W^{-1,p'}(\Omega)$. Then we can find $\psi_n \in L^{p'}(\Omega), \phi_n \in (L^{p'}(\Omega))^N$, such that $G_n - G = \psi_n + div\phi_n$ and $\|G_n - G\|_{W^{-1,p'}(\Omega)} = \max(\|\psi_n\|_{L^{p'}(\Omega)}, \|\phi_n\|_{(L^{P'}(\Omega))^N})$; then $\mu_0 = f + divg$ with $f = \xi + \varphi$ and $\mu_n^0 = f_n + divg_n$, with $f_n = \xi_n + \varphi + \psi_n$, $g_n = g + \phi_n$. Thus (μ_n) is a good approximation of μ, and satisfies (2.6). If μ is nonnegative, then μ_n is nonnegative.

(ii) We replace μ_n^0 by $\rho_m * \mu_n^0 = \rho_m * f_n + div(\rho_m * g_n)$, $m \in \mathbb{N}$, and observe that $\left|\rho_m * \mu_n^0\right|(\Omega) \leqq \left|\mu_n^0\right|(\Omega)$; then we can construct another sequence satisfying the conditions. ■

2.2 Locally renormalized solutions

Let $\mu \in \mathcal{M}(\Omega)$. Following the notion introduced in [6], we say that u is a **locally renormalized solution,** called **LR-solution**, of problem

$$-\Delta_p u = \mu, \qquad \text{in } \Omega, \tag{2.7}$$

if u is measurable and finite a.e. in Ω, $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, and

$$|u|^{p-1} \in L_{loc}^\sigma(\Omega), \forall\sigma \in [1, N/(N-p); \qquad |\nabla u|^{p-1} \in L_{loc}^\tau(\Omega), \forall\tau \in \tau \in [1, N/(N-1)), \tag{2.8}$$

and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support, such that $h(u)\varphi \in W^{1,p}(\Omega)$, there holds

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla(h(u)\varphi)dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^-. \tag{2.9}$$

Remark 2.9 *Hence the LR-solutions are solutions in $\mathcal{D}'(\Omega)$. From a recent result of [28], if $\mu \in \mathcal{M}^+(\Omega)$, any p-superharmonic function is a LR-solution, and conversely any LR-solution admits a p-superharmonic representant.*

3 Existence in the subcritical case

We first give a general existence result, where H satisfies some subcritical growth assumptions on u and ∇u, without any assumption on the sign of H or μ: we consider the problem

$$-\Delta_p u + H(x,u,\nabla u) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

where $\mu \in \mathcal{M}_b(\Omega)$. We say that u is a R-solution of problem (1.2) if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, and $H(x,u,\nabla u) \in L^1(\Omega)$ and u is a R-solution of

$$-\Delta_p u = \mu - H(x,u,\nabla u), \qquad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Theorem 3.1 *Let $\mu \in \mathcal{M}_b(\Omega)$, and assume that*

$$|H(x,u,\xi)| \leqq f(x)\,|u|^Q + g(x)\,|\xi|^q + \ell(x) \tag{3.2}$$

with $Q, q > 0$ and $f \in L^r(\Omega)$ with $Qr' < Q_c$, $g \in L^s(\Omega)$ with $qs' < q_c$, and $\ell \in L^1(\Omega)$.

Then there exists a R-solution of (3.1) if, either max$(Q,q) > p-1$ and $|\mu|(\Omega)$ and $\|\ell\|_{L^1(\Omega)}$ are small enough, or $q = p-1 > Q$ and $\|f\|_{L^r(\Omega)}$ is small enough, or $Q = p-1 > q$ and $\|g\|_{L^s(\Omega)}$ is small enough, or $q, Q < p-1$.

Proof. (i) Construction of a sequence of approximations. We consider a sequence $(\mu_n)_{n\geqq 1}$ of good approximations of μ, given in Lemma 2.8 (i). For any fixed $n \in \mathbb{N}^*$, and any $v \in W_0^{1,p}(\Omega)$ we define

$$M(v) = |\Omega|^{\frac{N-p}{N} - \frac{p-1}{Qr'}} \left(\int_\Omega |v|^{Qr'}\, dx \right)^{\frac{p-1}{Qr'}} + |\Omega|^{\frac{N-1}{N} - \frac{p-1}{qs'}} \left(\int_\Omega |\nabla v|^{qs'}\, dx \right)^{\frac{p-1}{qs'}},$$

$$\Phi_n(v)(x) = -\frac{H(x,v(x),\nabla v(x))}{1 + \frac{1}{n}(f(x)\,|v(x)|^Q + g(x)\,|\nabla v(x)|^q + \ell(x))}$$

so that $|\Phi_n(v)(x)| \leqq n$ a.e. in Ω. Let $\lambda > 0$ be a parameter. Starting from $u_1 \in W_0^{1,p}(\Omega)$ such that $M(u_1) \leqq \lambda$, we define $u_2 \in W_0^{1,p}(\Omega)$ as the solution of the problem

$$-\Delta_p u_2 = \Phi_1(u_1) + \mu_1 \quad \text{in } \Omega, \qquad U_2 = 0 \quad \text{on } \partial\Omega,$$

and by induction we define $u_n \in W_0^{1,p}(\Omega)$ as the solution of

$$-\Delta_p u_n = \Phi_{n-1}(u_{n-1}) + \mu_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

From (2.3), for any $\sigma \in (0, N/(N-p)$ and $\tau \in (0, N/(N-1))$,

$$|\Omega|^{\frac{N-p}{N} - \frac{1}{\sigma}} (\int_\Omega |u_n|^{(p-1)\sigma}\, dx)^{\frac{1}{\sigma}} + |\Omega|^{\frac{N-1}{N} - \frac{1}{\tau}} (\int_\Omega |\nabla u_n|^{(p-1)\tau}\, dx)^{\frac{1}{\tau}} \leqq C(\int_\Omega |\Phi_{n-1}(u_{n-1})|\, dx + 4\,|\mu|(\Omega)),$$

with $C = C(N,p,\sigma,\tau)$. We take $\sigma = Qr'/(p-1)$ and $\tau = qs'/(p-1)$; since

$$\int_\Omega |H(x,u_{n-1},\nabla u_{n-1})|\, dx \leqq \|f\|_{L^r(\Omega)} \int_\Omega |u_{n-1}|^{Qr'}\, dx)^{\frac{1}{r'}} + \|g\|_{L^s(\Omega)} (\int_\Omega |\nabla u_{n-1}|^{qs'}\, dx)^{\frac{1}{s'}} + \|\ell\|_{L^1(\Omega)} \tag{3.3}$$

we obtain

$$M(u_n) \leqq C(\int_\Omega |H(x, u_{n-1}, \nabla u_{n-1})|\, dx + 4\, |\mu|\, (\Omega)) \leqq b_1 M(u_{n-1})^{Q/(p-1)} + b_2 M(u_{n-1})^{q/(p-1)} + \eta + a$$

with $C = C(N, p, q, Q)$, and $b_1 = C\, \|f\|_{L^r(\Omega)}\, |\Omega|^{\frac{1}{r'} - \frac{Q}{Q_c}}$, $b_2 = C\, \|g\|_{L^s(\Omega)}\, |\Omega|^{1/s' - q/q_c}$, $\eta = C\, \|\ell\|_{L^1(\Omega)}$, $a = 4C\, |\mu|\, (\Omega)$. Then by induction, $M(u_n) \leqq \lambda$ for any $n \geqq 1$ if

$$b_1 \lambda^{Q/(p-1)} + b_2 \lambda^{q/(p-1)} + \eta + a \leqq \lambda. \tag{3.4}$$

When $Q < p - 1$ and $q < p - 1$, (3.4) holds for λ large enough. In the other cases, we note that it holds as soon as

$$b_1 \lambda^{Q/(p-1)-1} + b_2 \lambda^{q/(p-1)-1} \leqq 1/2, \quad \text{and } \eta \leqq \lambda/4, \; a \leqq \lambda/4. \tag{3.5}$$

First suppose that $Q > p - 1$ or $q > p - 1$. We take $\lambda \leqq 1$, small enough so that $(b_1^{Q/(p-1)} + b_2^{q/(p-1)})\lambda^{\max(Q,q)/(p-1)-1} \leqq 1/2$, and then $\eta, a \leqq \lambda/4$. Next suppose for example that $Q = p-1 > q$, a is arbitrary. If b_1 small enough, and η, a are arbitrary, then we obtain (3.5) for λ large enough.

(ii) Convergence: Since $M(u_n) \leqq \lambda$, in turn from (3.3), $(H(x, u_n, \nabla u_n))$ is bounded in $L^1(\Omega)$, and then also $\Phi_n(u_n)$. Thus

$$\int_\Omega |\Phi_{n-1}(u_{n-1})|\, dx + |\mu_n|\, (\Omega) \leqq C_\lambda := b_1 \lambda^{Q/(p-1)} + b_2 \lambda M^{q/(p-1)} + \eta + 4\, |\mu|\, (\Omega).$$

From Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u, (∇u_n) converges *a.e.* to ∇u, and $\left(u_n^{p-1}\right)$ converges strongly in $L^\sigma(\Omega)$, for any $\sigma \in [1, N/(N-p))$, and finally $\left(|\nabla u_n|^{p-1}\right)$ converges strongly in $L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$. Therefore $(u_n^{Qr'})$ and $(|\nabla u_n|^{qr'})$ converge strongly in $L^1(\Omega)$, in turn $(\Phi_n(x, u_n, \nabla u_n))$ converges strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$. Then $(\Phi_n(x, u_n, \nabla u_n) + \mu_n)$ is a sequence of good approximations of $H(x, u, \nabla u) + \mu$. From Theorem 2.6, u is a R-solution of problem (3.1). ■

Remark 3.2 *Our proof is not based on the Schauder fixed point theorem, so we do not need that* $1 \leqq Qr'$ *or* $1 \leqq qs'$. *Hence we improve the former result of [19] for problem (1.5) where* H *only depends on* u, *proved for* $1 \leqq Qr'$, *implying* $1 < Q_c$. *Here we have no restriction on* Q_c *and* q_c.

Next we consider the case where H and μ are nonnegative; then we do not need that the data are small:

Theorem 3.3 *Consider the problem (3.1)*

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.6}$$

where $\mu \in \mathcal{M}_b^+(\Omega)$, *and*

$$0 \leqq H(x, u, \xi) \leqq C(|u|^Q + |\xi|^q) + \ell(x), \tag{3.7}$$

with $0 < Q < Q_c, 0 < q < q_c, C > 0, \ell \in L^1(\Omega)$. *Then there exists a nonnegative R-solution of problem (3.6).*

Proof. We use the good approximation of μ by a sequence of measures $\mu_n = \mu_n^0 + \lambda_n$, with $\mu_n^0 \in \mathcal{D}^+(\Omega), \lambda_n \in C_b^+(\Omega)$, given at Lemma 2.8 (ii). Then there exists a weak nonnegative solution $u_n \in W_0^{1,p}(\Omega)$ of the problem

$$-\Delta_p u_n + H(x, u_n, \nabla u_n) = \mu_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Indeed 0 is a subsolution, and the solution $\psi_n \in W_0^{1,p}(\Omega)$ of $-\Delta_p \psi_n = \mu_n$ in Ω, is a supersolution. Since $\mu_n \in L^\infty(\Omega)$, there holds $\psi \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, thus $\psi \in W^{1,\infty}(\Omega)$. From [12, Theorem 2.1], since $Q_c \leqq p$, there exists a weak solution $u_n \in W_0^{1,p}(\Omega)$, such that $0 \leq u_n \leq \psi_n$, hence $u_n \in L^\infty(\Omega)$, and $u_n \in W_{loc}^{1,r}(\Omega)$ for some $r > p$. Taking $\varphi = k^{-1} T_k(u_n - m)$ with $m \geq 0$, $k > 0$, as a test function, we get from (2.6)

$$\frac{1}{k} \int_{\{m \leqq u \leqq m+k\}} |\nabla u_n|^p \, dx \leq \mu_n(\Omega) \leq 4\mu(\Omega), \tag{3.8}$$

then from Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u, $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$and (∇u_n) converges *a.e.* to ∇u, and $(|\nabla u_n|^p)$, $\left(u_n^{p-1}\right)$ converges strongly in $L^\sigma(\Omega)$ for any $\sigma \in [1, N/(N-p))$, $\left(|\nabla u_n|^{p-1}\right)$ converges strongly in $L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$. Then $(u_n^{Qr'})$ and $(|\nabla u_n|^{qr'})$ converge strongly in $L^1(\Omega)$, in turn $(H(x, u_n, \nabla u_n))$ converges strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$. Applying Theorem 2.6 to $\mu_n - H(x, u_n, \nabla u_n)$ as above, we still obtain that u is a R-solution of (3.6). ■

4 Necessary conditions for existence and removability results

Let $\mu \in \mathcal{M}(\Omega)$. We consider the local solutions of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \qquad \text{in } \Omega, \tag{4.1}$$

We say that u is a **weak solution** of (4.1) if u is measurable and finite a.e. in Ω, $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, $H(x, u, \nabla u) \in L_{loc}^1(\Omega)$ and (4.1) holds in $\mathcal{D}'(\Omega)$. We say that u is a **LR-solution** of (4.1) if $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, and $|\nabla u|^q \in L_{loc}^1(\Omega)$ and u is a LR-solution of

$$-\Delta_p u = \mu - H(x, u, \nabla u), \qquad \text{in } \Omega.$$

Remark 4.1 *If $q \geqq 1$ and u is a weak solution, then u satisfies (2.8), see for example [31, Lemma 2.2 and 2.3], thus $u \in W_{loc}^{1,q}(\Omega)$.*

Lemma 4.2 *Let $\mu \in \mathcal{M}(\Omega)$. Assume that (4.1) admits a weak solution u.*
(i) If

$$|H(x, u, \xi)| \leqq C_1 |\xi|^q + \ell(x) \tag{4.2}$$

with $C_1 > 0$ and $\ell \in L^1(\Omega)$, then setting $C_2 = C_1 + q_* - 1$, for any $\zeta \in \mathcal{D}^+(\Omega)$,

$$\left| \int_\Omega \zeta^{q_*} d\mu \right| \leq C_2 \int_\Omega |\nabla u|^q \, \zeta^{q_*} dx + \int_\Omega |\nabla \zeta|^{q_*} \, dx + \int_\Omega \ell \zeta^{q_*} dx. \tag{4.3}$$

(ii) If H has a constant sign, and

$$C_0 \left|\xi\right|^q - \ell(x) \leqq \left|H(x,u,\xi)\right|, \tag{4.4}$$

then for some $C = C(C_0, p, q)$,

$$\int_\Omega \left|\nabla u\right|^q \zeta^{q_*} dx \leqq C(\left|\int_\Omega \zeta^{q_*} d\mu\right| + \int_\Omega \left|\nabla\zeta\right|^{q_*} dx + \int_\Omega \ell\zeta^{q_*} dx) \tag{4.5}$$

Proof. By density, we can take ζ^{q_*} as a test function, and get

$$\int_\Omega \zeta^{q_*} d\mu = -\int_\Omega H(x,u,\nabla u)\zeta^{q_*} dx + q_* \int_\Omega \left|\nabla u\right|^{p-2} \nabla u.\zeta^{q_*-1}\nabla\zeta dx;$$

and from the Hölder inequality, for any $\varepsilon > 0$,

$$q_* \int_\Omega \left|\nabla u\right|^{p-1} \zeta^{q_*-1} \left|\nabla\zeta\right| dx \leqq (q_* - 1)\varepsilon \int_\Omega \left|\nabla u\right|^q \zeta^{q_*} dx + \varepsilon^{1-q_*} \int_\Omega \left|\nabla\zeta\right|^{q_*} dx \tag{4.6}$$

which implies (4.3). If H has a constant sign, then

$$\begin{aligned} C_0 \int_\Omega \left|\nabla u\right|^q \zeta^{q_*} dx - \int_\Omega \ell dx &\leqq \int_\Omega \left|H(x,u,\nabla u)\right| \zeta^{q_*} dx = \left|\int_\Omega H(x,u,\nabla u)\zeta^{q_*} dx\right| \\ &\leqq \left|\int_\Omega \zeta^{q_*} d\mu\right| + q_* \int_\Omega \left|\nabla u\right|^{p-1} \zeta^{q_*-1} \left|\nabla\zeta\right| dx, \end{aligned}$$

thus (4.5) follows after taking ε small enough. ■

Proposition 4.3 *Let $\mu \in \mathcal{M}(\Omega)$, and assume that (4.1) admits a weak solution u.*

(i) If (4.2) holds, then $\mu \in \mathcal{M}^{q_*}(\Omega)$.

(ii) If $H(x,u,\xi) \leqq -C_0 \left|\xi\right|^q$ and μ and u are nonnegative, then in addition there exists $C = C(C_0, p, q) > 0$ such that for any compact $K \subset \Omega$,

$$\mu(K) \leqq C cap_{1,q_*}(K, \Omega). \tag{4.7}$$

Proof. (i) Let E be a Borel set such that $cap_{1,q_*}(E, \Omega) = 0$. There exist two measurable disjoint sets A, B such that $\Omega = A \cup B$ and $\mu^+(B) = \mu^-(A) = 0$. Let us show that $\mu^+(A \cap E) = 0$. Let K be any fixed compact set in $A \cap E$. Since $\mu^-(K) = 0$, for any $\delta > 0$ there exists a regular domain $\omega \subset\subset \Omega$ containing K, such that $\mu^-(\omega) < \delta$. Then there exists $\zeta_n \in \mathcal{D}(\omega)$ such that $0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ on a neighborhood of K contained in ω, and (ζ_n) converges to in $W^{1,q_*}(\mathbb{R}^N)$ and *a.e.* in Ω, see [2]. There holds

$$\mu^+(K) \leq \int_\omega \zeta_n^{q_*} d\mu^+ = \int_\omega \zeta_n^{q_*} d\mu + \int_\omega \zeta_n^{q_*} d\mu^- \leq \int_\omega \zeta_n^{q_*} d\mu + \delta$$

and from (4.3),

$$\left|\int_\Omega \zeta_n^{q_*} d\mu\right| \leq C_2 \int_\Omega \left|\nabla u\right|^q \zeta_n^{q_*} dx + \int_\Omega \left|\nabla\zeta_n\right|^{q_*} dx + \int_\Omega \ell\zeta_n^{q_*} dx$$

And $\lim_{n\to\infty}\int_\Omega |\nabla u|^q \zeta_n^{q_*}dx = 0$, from the dominated convergence theorem, thus $\left|\int_\Omega \zeta_n^{q_*}d\mu\right| \leq \delta$ for large n; then $\mu^+(K) \leq 2\delta$ for any $\delta > 0$, thus $\mu^+(K) = 0$, hence $\mu^+(A\cap E) = 0$; similarly we get $\mu^-(B\cap E) = 0$, hence $\mu(E) = 0$.
(ii) Here we find

$$\int_\Omega \zeta^{q_*}d\mu + C_0\int_\Omega |\nabla u|^q \zeta^{q_*}dx \leqq q_*\int_\Omega |\nabla u|^{p-2}\nabla u.\zeta^{q_*-1}\nabla\zeta dx$$

and hence from (4.6) with $\varepsilon > 0$ small enough, for some $C = C(C_0, p, q)$,

$$\int_\Omega \zeta^{q_*}d\mu \leq C\int_\Omega |\nabla\zeta|^{q_*}dx$$

and (4.7) follows, see [34]. ∎

Remark 4.4 *Property (ii) extends the results of [20] and [39, Theorem 3.1] for equation (1.4).*

Next we show a removability result:

Theorem 4.5 *Assume that H has a constant sign and satisfies (4.2) and (4.4). Let F be any relatively closed subset of Ω, such that $cap_{1,q_*}(F,\mathbb{R}^N) = 0$, and $\mu \in \mathcal{M}^{q_*}(\Omega)$.*

(i) Let $1 < q \leqq p$. Let u be any LR-solution of

$$-\Delta_p u + H(x,u,\nabla u) = \mu \qquad in\ \Omega\backslash K \tag{4.8}$$

Then u is a LR-solution of

$$-\Delta_p u + H(x,u,\nabla u) = \mu \qquad in\ \Omega. \tag{4.9}$$

(ii) Let $q > p$ and u be a weak solution of (4.8), then u is a weak solution of (4.9).

Proof. (i) Let $1 < q \leqq p$. From our assumption, $T_k(u) \in W^{1,p}_{loc}(\Omega\backslash F)$, for any $k > 0$, and $|u|^{p-1} \in L^\sigma_{loc}(\Omega)$, for any $\sigma \in [1, N/(N-p))$, and $|\nabla u|^{p-1} \in L^\tau_{loc}(\Omega\backslash F)$, for any $\tau \in [1, N/(N-1))$, and $|\nabla u|^q \in L^1_{loc}(\Omega\backslash F)$. For any compact $K \subset \Omega$, there holds $cap_{1,p}(F\cap K,\mathbb{R}^N) = 0$, because $p \leqq q_*$, thus $T_k(u) \in W^{1,p}_{loc}(\Omega)$, see [21, Theorem 2.44]. And u is measurable on Ω and finite a.e. in Ω, thus we can define ∇u a.e. in Ω by the formula $\nabla u(x) = \nabla T_k(u)(x)$ a.e. on the set $\{x \in \Omega : |u(x)| \leqq k\}$.

Let us consider a fixed function $\zeta \in \mathcal{D}^+(\Omega)$ and let $\omega \subset\subset \Omega$ such that supp$\zeta \subset \omega$ and set $K_\varsigma = F\cap$ suppζ. Then K_ς is a compact and $cap_{1,q_*}(K,\mathbb{R}^N) = 0$. Thus there exists $\zeta_n \in \mathcal{D}(\omega)$ such that $0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ on a neighborhood of K contained in ω, and (ζ_n) converges to 0 in $W^{1,q_*}(\mathbb{R}^N)$; we can assume that the convergence holds everywhere on $\mathbb{R}^N\backslash N$, where $cap_{1,q_*}(N,\mathbb{R}^N) = 0$, see for example [4, Lemmas 2.1,2.2]. From Lemma 4.2 applied to $\xi_n = \zeta(1-\zeta_n)$ in $\Omega\backslash F$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^q \xi_n^{q_*}dx &\leqq C(\int_\Omega \xi_n^{q_*}d|\mu| + \int_\Omega |\nabla\xi_n|^{q_*}dx + \int_\Omega \ell\xi_n^{q_*}dx)\\ &\leqq C(\int_\Omega \zeta^{q_*}d|\mu| + \int_\Omega |\nabla\zeta|^{q_*}dx + \int_\Omega |\nabla\zeta_n|^{q_*}dx + \int_\Omega \ell\zeta^{q_*}dx).\end{aligned} \tag{4.10}$$

From the Fatou Lemma, we get $|\nabla u|^q \zeta^{q_*} \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^q \zeta^{q_*}dx \leqq C_\zeta := C(\int_\Omega \zeta^{q_*}d|\mu| + \int_\Omega |\nabla\zeta|^{q_*}dx\int_\Omega \ell\zeta^{q_*}dx), \tag{4.11}$$

where C_ζ also depends on ζ. Taking $T_k(u)\xi_n^{q_*}$ as test function, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla(T_k(u))|^p \xi_n^{q_*} dx + \int_\Omega H(x,u,\nabla u) T_k(u)\xi_n^{q_*} dx \\
= &\int_\Omega T_k(u)\xi_n^{q_*} d\mu_0 + k(\int_\Omega \xi_n^{q_*} d\mu_s^+ + \int_\Omega \xi_n^{q_*} d\mu_s^-) + \int_\Omega T_k(u) |\nabla u|^{p-2} \nabla u.\nabla(\xi_n^{q_*})dx;
\end{aligned}$$

From the Hölder inequality, we deduce

$$\begin{aligned}
&\frac{1}{k}\left|\int_\Omega T_k(u) |\nabla u|^{p-2} \nabla u.\nabla(\xi_n^{q_*})dx\right| \\
\leqq\ & q_*(\int_\Omega \zeta^{q_*-1} |\nabla u|^{p-1} |\nabla\zeta|)dx + \int_\Omega \zeta^{q_*} |\nabla u|^{p-1} |\nabla\zeta_n| \, dx) \\
\leqq\ & (q_*-1)\int_\Omega |\nabla u|^q \zeta^{q_*} dx + \int_\Omega |\nabla\zeta|^{q_*} dx + q_*(\int_\Omega |\nabla u|^q \zeta^{q_*} dx + \int_\Omega \zeta^{q_*} |\nabla\zeta_n|^{q_*} dx) \\
\leqq\ & 2q_* C_\zeta + \int_\Omega |\nabla\zeta|^{q_*} dx + o(n).
\end{aligned}$$

Thus from (4.2), with a new constant C_ζ,

$$\int_\Omega |\nabla(T_k(u))|^p \xi_n^{q_*} dx \leqq (k+1)C_\zeta + o(n);$$

hence from the Fatou Lemma,

$$\int_\Omega |\nabla(T_k(u))|^p \zeta^{q_*} dx \leqq (k+1)C_\zeta.$$

Therefore $|u|^{p-1} \in L^\sigma_{loc}(\Omega)$, $\forall\ \sigma \in [1, N/(N-p))$ and $|\nabla u|^{p-1} \in L^\tau_{loc}(\Omega), \forall\ \tau \in [1, N/(N-1))$, from a variant of the estimates of [5] and [10], see [37, Lemma 3.1].

Finally we show that u is a LR-solution in Ω : let $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support in Ω, such that $h(u)\varphi \in W^{1,p}(\Omega)$; let $\omega \subset\subset \Omega$ such that supp$\zeta \subset \omega$ and set $K = F \cap$suppζ, and consider $\zeta_n \in \mathcal{D}(\mathbb{R}^N)$ as above; then $(1-\zeta_n)\varphi \in W^{1,m}(\Omega\backslash F)$ and $h(u)(1-\zeta_n)\varphi \in W^{1,p}(\Omega\backslash F)$ and has a compact support in $\Omega\backslash F$, then we can write

$$I_1 + I_2 + I_3 + I_4 = \int_\Omega h(u)\varphi(1-\zeta_n)d\mu_0 + h(+\infty)\int_\Omega \varphi(1-\zeta_n)d\mu_s^+ - h(-\infty)\int_\Omega \varphi(1-\zeta_n)d\mu_s^-,$$

with

$$\begin{aligned}
I_1 &= \int_\Omega |\nabla u|^{p-2} \nabla u.h'(u)\varphi(1-\zeta_n)dx, \qquad I_2 = -\int_\Omega |\nabla u|^{p-2} \nabla u.h(u)\varphi\nabla\zeta_n dx \\
I_3 &= \int_\Omega |\nabla u|^{p-2} \nabla u.h(u)(1-\zeta_n)\nabla\varphi dx, \qquad I_4 = \int_\Omega H(x,u,\nabla u)h(u)\varphi(1-\zeta_n)dx.
\end{aligned}$$

We can go to the limit in I_1 as $n \to \infty$, from the dominated convergence theorem, since there exists $a > 0$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u.h'(u)\varphi(1-\zeta_n)dx = \int_\Omega |\nabla T_a(u)|^{p-2} \nabla T_a(u).h'(T_a(u))\varphi(1-\zeta_n)dx.$$

Furthermore $I_2 = o(n)$, because

$$\left| \int_\Omega |\nabla u|^{p-2} \nabla u.h(u)\varphi \nabla \zeta_n dx \right| \leqq \|h\|_{L^\infty(\mathbb{R})} (\int_\Omega |\nabla u|^q \varphi dx)^{1/q} \|\nabla \zeta_n\|_{L^{q_*}(\mathbb{R}^N)};$$

we can go to the limit in I_3 because $|\nabla \varphi| \in L^m(\Omega)$ and $|\nabla u|^{p-1} \in L^\tau_{loc}(\Omega), \forall \tau \in [1, N/(N-1))$; in I_4 from (4.11) and (4.2), and in the right hand side because $h(u)\varphi \in L^1(\Omega, d\mu_0)$, see [16, Remark 2.26] and $\zeta_n \to 0$ everywhere in $\mathbb{R}^N \backslash N$ and $\mu(N) = 0$. Then we conclude:

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla(h(u)\varphi) dx + \int_\Omega H(x, u, \nabla u) h(u)\varphi dx \\ = &\int_\Omega h(u)\varphi d\mu_0 + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-. \end{aligned}$$

(ii) Assume that $q > p > 1$ (hence $1 < q_* < p$) and u is a weak solution in $\Omega \backslash F$. Then $u \in W^{1,q}_{loc}(\Omega \backslash F)$ implies $u \in W^{1,q_*}_{loc}(\Omega \backslash F) = W^{1,q_*}_{loc}(\Omega)$, hence $|\nabla u|$ is well defined in $L^1_{loc}(\Omega)$. As in part (i) we obtain that $|\nabla u|^q \zeta^{q_*} \in L^1(\Omega)$, hence $|\nabla u|^q \in L^1_{loc}(\Omega)$. For any $\varphi \in \mathcal{D}(\Omega)$, and ω containing suppφ, we have $\varphi(1 - \zeta_n) \in \mathcal{D}(\Omega \backslash F)$, then we can write $J_1 + J_2 + J_3 = \int_\Omega \varphi(1 - \zeta_n) d\mu$, with

$$J_1 = \int_\Omega (1-\zeta_n) |\nabla u|^{p-2} \nabla u.\nabla \varphi dx, \ J_2 = -\int_\Omega \varphi |\nabla u|^{p-2} \nabla u.\nabla \zeta_n dx, \ J_3 = \int_\Omega H(x, u, \nabla u)\varphi(1-\zeta_n) dx.$$

Now we can go to the limit in J_1 and J_3 from the dominated convergence theorem, because $|\nabla u|^q \in L^1_{loc}(\Omega)$ and $q > p - 1$; and $(\int_\Omega \varphi(1 - \zeta_n) d\mu)$ converges to $\int_\Omega \varphi d\mu$ as above. And J_2 converges to 0, because $|\nabla u|^{p-1} \in L^{q/(p-1)}_{loc}(\Omega)$ and $|\nabla \zeta_n|$ tends to 0 in $L^{q_*}(\Omega)$. Then u is a weak solution in Ω. ■

5 Existence in the supercritical case

Here the problem is delicate and many problems are still unsolved.

5.1 Case of a source term

Here we consider problem

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

The main question is the following:

If $\mu \in M^{q_*}_b(\Omega)$ satisfies condition (4.7) with a constant $C > 0$ small enough, does (5.1) admit a solution?

In the case $p = 2 < q$, the problem has been solved in [20]. In that case one can define the solutions in a very weak sense. According to [14], setting $\rho(x) = dist(x, \partial\Omega)$, a function u is called a *very weak solution* of (5.1) if $u \in W^{1,q}_{loc}(\Omega) \cap L^1(\Omega)$, $|\nabla u|^q \in L^1(\Omega, \rho dx)$ and for any $\varphi \in C^2(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$,

$$-\int_\Omega u \Delta \varphi dx = \int_\Omega |\nabla u|^q \varphi dx + \int_\Omega \varphi d\mu.$$

Theorem 5.1 ([20]) *Let $\mu \in \mathcal{M}^+(\Omega)$. If $1 < q$ and $p = 2$ and (5.1) has a very weak solution, then*

$$\mu(K) \leqq C cap_{1,q'}(K, \Omega) \tag{5.2}$$

for any compact $K \subset \Omega$, and some $C < C_1(N,q)$. Conversely, if $2 < q$ and (5.2) holds for some $C < C_2(N,q,\Omega)$ then (5.1) has a very weak nonnegative solution.

In the general case $p > 1$, such a notion of solution does not exist. The problem (5.1) with $p < q$ was studied by [39] for signed measures $\mu \in \mathcal{M}_b(\Omega)$ such that

$$[\mu]_{1,q^*,\Omega} = \sup \left\{ \frac{|\mu(K \cap \Omega)|}{Cap_{1,q^*}(K, \mathbb{R}^N)} : K \text{ compact of } \mathbb{R}^N, Cap_{1,q^*}(K, \mathbb{R}^N) > 0 \right\} < \infty.$$

Theorem 5.2 ([39]) *Let $1 < p < q$. Let $\mu \in \mathcal{M}_b(\Omega)$. There exists $C_1 = C_1(N,p,q,\Omega)$ such that if*

$$|\mu(K \cap \Omega)| \leqq C cap_{1,q^*}(K, \mathbb{R}^N) \tag{5.3}$$

for any compact $K \subset \mathbb{R}^N$, and some $C < C_1$, then (5.1) has a weak solution $u \in W_0^{1,q}(\Omega)$, such that $[|\nabla u|^q]_{1,q^*,\Omega}$ is finite. In particular this holds for any $\mu \in L^{N/q^*,\infty}(\Omega)$.

Very recently the case $p = q$, has been studied in [25] for signed measures satisfying a trace inequality: setting $p^\# = (p-1)^{2-p}$ if $p \geqq 2$, $p^\# = 1$ if $p < 2$, they show in particular the following:

Theorem 5.3 ([25]) *Let $1 < p = q$. Let $\mu \in \mathcal{M}_b(\Omega)$ such that*

$$-C_1 \int_\Omega |\nabla \zeta|^p \, dx \leqq \int_\Omega |\zeta|^p \, d\mu \leq C_2 \int_\Omega |\nabla \zeta|^p \, dx, \qquad \forall \zeta \in \mathcal{D}(\Omega), \tag{5.4}$$

with $C_1 > 0$ and $C_2 \in (0, p^\#)$. Then (5.1) has a weak solution $u \in W^{1,p}_{loc}(\Omega)$.

The existence for problem (5.1) is still open in the case $q < p$ for $p \neq 2$

5.2 Case of an absorption term

Here we consider problem (1.2) in case of absorption, where $\mu \in M_b^+(\Omega)$ and we look for a nonnegative solution. In the model case

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.5}$$

the main question is the following: *If $\mu \in M_b^{q_*+}(\Omega)$, hence $\mu = f + div g$, with $f \in L^1(\Omega)$ and $g \in (L^{q/(p-1)}(\Omega))^N$, does (5.5) admits a nonnegative solution?*

Remark 5.4 *Up to changing u into $-u$, the results of Theorem 5.2 and 5.3 are also available for the problem (5.5) but we have* ***no information on the sign of*** *u.*

In the sequel we give two partial results of existence.

5.2.1 Case $q \leqq p$ and $\mu \in \mathcal{M}_b^{p+}(\Omega)$

Here we assume that $\mu \in \mathcal{M}_b^{p+}(\Omega)$, subspace of $\mathcal{M}_b^{q_*+}(\Omega)$. Our proof is directly inspired from the results of [11] for the problem (3.6), where $q = p$ and $H(x,u,\xi)u \geqq 0$.

Theorem 5.5 *Let $p-1 < q \leqq p$. Let $\mu \in \mathcal{M}_b^{p+}(\Omega)$, and*

$$\begin{aligned} 0 &\leqq H(x,u,\xi) \leqq C_1 |\xi|^p + \ell(x), && (5.6)\\ H(x,u,\xi) &\geqq C_0 |\xi|^q \text{ for } u \geqq L, && (5.7)\end{aligned}$$

with $\ell(x) \in L^1(\Omega), C_k, C_0, L \geqq 0$. Then there exists a nonnegative R-solution of problem (1.2).

Remark 5.6 *The result was known in the case where $H(x,u,\nabla u) = |\nabla u|^q$, $p = 2$, and $\mu \in L^1(\Omega)$ (see for example [1], where the existence for any $\mu \in \mathcal{M}_b^{2+}(\Omega)$ is also claimed, without proof). For $p \neq 2$, the case $q < p$, $\mu \in L^1(\Omega)$ is partially treated in [38].*

Proof. Let $\mu = f - divg$ with $f \in L^{1+}(\Omega)$ and $g = (g_i) \in (L^{p'}(\Omega))^N$. Here again we use the good approximation of μ by a sequence of measures $\mu_n \in \mathcal{M}_b^+(\Omega)$ given at Lemma 2.8 (ii), $\lambda_n = 0$, thus $\mu_n = \mu_n^0 = f_n - divg_n$, with $f_n \in \mathcal{D}^+(\Omega)$ and $g_n = (g_{n,i}) \in (\mathcal{D}(\Omega)^N)$. Hence there exists a weak nonnegative solution $u_n \in W_0^{1,p}(\Omega)$ of the problem

$$-\Delta_p u_n + H(x,u_n,\nabla u_n) = \mu_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Since $H(x,u,\xi) \geqq 0$, taking $\varphi = k^{-1}T_k(u_n - m)$ with $m \geq 0$, $k > 0$, as a test function, we still obtain (3.8). From Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u, $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$, (∇u_n) converges *a.e.* to ∇u, and $\left(u_n^{p-1}\right)$ converges strongly in $L^\sigma(\Omega)$, for any $\sigma \in [1, N/(N-p))$. Thus $\lim_{k\to\infty} \sup_{n\in\mathbb{N}} |\{u_n > k\}| = 0$, and $\left(|\nabla u_n|^{p-1}\right)$ converges strongly in $L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$. Moreover the choice of φ with $m + k > L$ gives

$$\frac{1}{k}\int_{\{m \leqq u \leqq m+k\}} |\nabla u_n|^p\,dx + C_0 \int_{\{u_n \geq m+k\}} |\nabla u_n|^q\,dx \leqq \mu_n(\Omega) \leqq 4\mu(\Omega).$$

Taking $m = 0$ we obtain

$$\int_\Omega |\nabla u_n|^q\,dx \leq \int_{\{u_n \geq k\}} |\nabla u_n|^q\,dx + \int_\Omega |\nabla T_k(u_n)|^q\,dx \leq 4C_0^{-1}\mu(\Omega) + \int_\Omega |\nabla T_k(u_n)|^p\,dx + |\Omega|$$

since $q \leqq p$; thus from the Fatou Lemma, $|\nabla u|^q \in L^1(\Omega)$. Moreover using $\varphi = T_1(u_n - k)$,

$$\int_{\{k-1 \leqq u_n \leqq k\}} |\nabla u_n|^p\,dx + \int_{\{u_n \geq k\}} H(x,u_n,\nabla u_n)dx \leqq \int_{\{u_n \geq k-1\}} f_n dx + \int_{\{k-1 \leq u_n \leq k\}} |g_n.\nabla u_n|\,dx.$$

Therefore, from the Hölder inequality,

$$\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p\,dx + p' \int_{\{u_n \geq k\}} H(x,u_n,\nabla u_n)dx \leq \int_{\{u_n \geq k-1\}} f_n dx + (\sum_{i=1}^N \int_{\{k-1 \leq u_n \leq k\}} |g_{n,i}|^{p'}\,dx).$$

From Lemma 2.8, there holds

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}(\int_{\{k-1\leq u_n\leq k\}}|\nabla u_n|^p\,dx+\int_{\{u_n\geq k\}}H(x,u_n,\nabla u_n)dx)=0. \tag{5.8}$$

Next we prove the strong convergence of the truncates in $W_0^{1,p}(\Omega)$ as in [11]: we take as test function

$$\varphi_n=\Phi(T_k(u_n)-T_k(u)),\text{ where }\Phi(s)=se^{\theta^2s^2/4},$$

where $\theta>0$ will be chosen after, thus $\Phi'(s)\geq\theta\,|\Phi(s)|+1/2$. Then $\varphi_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and we have $|\varphi_n|\leq\Phi(k)$; setting $\psi_n=\Phi'(T_k(u_n)-T_k(u))$, we have $0\leq\psi_n\leq\Phi'(k)$. Then $\varphi_n\to0$, $\psi_n\to1$ in $L^\infty(\Omega)$ weak $*$ and *a.e.* in Ω. We set $a(\xi)=|\xi|^{p-2}\xi$, and

$$X=\int_\Omega(a(\nabla(T_k(u_n))-a(\nabla(T_k(u))).\nabla(T_k(u_n)-T_k(u))\psi_ndx,$$

and get

$$X+I_1=I_2+I_3+I_4,$$

with

$$I_1=\int_\Omega H(x,u_n,\nabla u_n)\varphi_ndx,\qquad I_2=\int_\Omega a(\nabla(T_k(u)).\nabla(T_k(u)-T_k(u_n))\psi_ndx,$$

$$I_3=\int_\Omega f_n\varphi_ndx+\int_\Omega div(g_n-g)\varphi_ndx+\int_\Omega g.\nabla(T_k(u_n)-T_k(u))\psi_ndx,$$

$$I_4=-\int_\Omega a(\nabla(u_n-T_k(u_n)).\nabla(T_k(u_n)-T_k(u))\psi_ndx=\int_{\{u_n\geq k\}}a(\nabla(u_n-T_k(u_n)).\nabla(T_k(u))\psi_ndx.$$

One can easily see that $|I_2|+|I_3|+|I_4|=o(n)$. Since $H(x,u_n,\nabla u_n)\geqq0$ for $u_n\geq k$, then $X\leqq I_5+o(n)$, where

$$I_5=\left|\int_{\{u_n<k\}}H(x,u_n,\nabla u_n)\varphi_ndx\right|\leq C_1\int_\Omega|\nabla(T_ku_n)|^p)\,|\varphi_n|\,dx+\int_\Omega l\,|\varphi_n|\,dx\leqq C_1(Y+I_7)+o(n),$$

with

$$Y=\int_\Omega(a(\nabla(T_k(u_n))-a(\nabla(T_k(u))).\nabla(T_k(u_n)-T_k(u))\,|\varphi_n|\,dx,$$

$$I_7=\int_\Omega a(\nabla(T_k(u))).\nabla(T_k(u_n)-T_k(u))\,|\varphi_n|\,dx+\int_\Omega(a(\nabla(T_k(u_n)).\nabla(T_k(u))\,|\varphi_n|\,dx$$

and then $I_7=o(n)$. We get finally $X\leqq C_1Y+o(n)$; choosing $\theta=2C_1$, we deduce that

$$\int_\Omega(a(\nabla(T_k(u_n))-a(\nabla(T_k(u))).\nabla(T_k(u_n)-T_k(u))dx=o(n).$$

Hence $(T_k(u_n))$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$. Therefore $H(x,u_n,\nabla u_n)$ is equi-integrable, from (5.6) and (5.8), since for any measurable set $E\subset\Omega$,

$$\int_E H(x,u_n,\nabla u_n)dx\leqq C_1\int_E|\nabla(T_ku_n)|^p\,dx+\int_E\ell dx+\int_{\{u_n\geq k\}}H(x,u_n,\nabla u_n)dx.$$

Then $(H(x,u_n,\nabla u_n))$ converges to $H(x,u,\nabla u)$ strongly in $L^1(\Omega)$; thus $(\mu_n - H(x,u_n,\nabla u_n))$ is a good approximation of $\mu - H(x,u,\nabla u)$, and u is a R-solution of problem (3.1) from Theorem 2.6. ■

Remark 5.7 *In the case $p-1<q<p$, and if (5.6) is replaced by*

$$0 \leqq H(x,u,\xi) \leqq C_1 |\xi|^q + \ell(x), \tag{5.9}$$

the proof is much shorter: in order to prove the equi-integrability of $(H(x,u_n,\nabla u_n))$ we do not need to prove the strong convergence of the truncates: indeed for any measurable set $E \subset \Omega$,

$$\int_E |\nabla u_n|^q \,dx \leqq \int_E |\nabla (T_k u_n)|^q \,dx + \int_{\{u_n \geq k\}} |\nabla u_n|^q \,dx$$

and $(\nabla T_k(u_n))$ converges strongly to $\nabla T_k(u)$ in $L^q(\Omega)$ and (5.8) holds. Then $(H(x,u_n,\nabla u_n))$ converges to $H(x,u,\nabla u)$ strongly in $L^1(\Omega)$.

5.2.2 Case where μ satisfies (4.7)

Here we assume that $\mu \in \mathcal{M}_b^+(\Omega)$ satisfies a capacity condition of type (4.7). For simplicity we assume that μ has a compact support in Ω. In the sequel we prove the following:

Theorem 5.8 *Let $1<q\leqq p$ or $p=2$. Assume that $\mu \in \mathcal{M}_b^+(\Omega)$, has a compact support and satisfies*

$$\mu(K) \leqq C_1 cap_{1,q^*}(K,\Omega), \qquad \text{for any compact } K \subset \Omega, \tag{5.10}$$

for some $C_1 = C(N,q,\Omega) > 0$ (non necessarily small). Then there exists a nonnegative R-solution u of problem (5.5), such that $[|\nabla u|^q]_{1,q^*,\Omega}$ is finite.

First recall some equivalent properties of measures, see [35, Theorem 1.2], [20, Lemma 3.3], see also [39]:

Remark 5.9 *1) Let $\mu \in \mathcal{M}_b^+(\Omega)$, extended by 0 to $\mathbb{R}^N$. Then (5.10) holds if and only if there exists $C_2>0$ such that*

$$\int_\Omega \zeta^{q_*} d\mu \leq C_2 \int_\Omega |\nabla \zeta|^{q_*} \,dx, \qquad \forall \zeta \in \mathcal{D}^+(\Omega); \tag{5.11}$$

the constants of equivalence between C_1, C_2 only depend on N, q_*, Ω.

If moreover μ has a compact support $K_0 \subset \Omega$, then (5.10) holds if and only if there exists $C_3>0$ such that

$$\mu(K) \leqq C_3 Cap_{1,q_*}(K,\mathbb{R}^N) \qquad \text{for any compact } K \subset \mathbb{R}^N; \tag{5.12}$$

the constants of equivalence between C_1, C_3 only depend on N, q_*, K_0.

2) Let $\nu \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then (5.12) holds if and only if there exists $C_4>0$ such that $J_1(\nu)$ is finite a.e. and

$$J_1((J_1(\nu))^{q_*}) \leqq C_4 J_1(\nu) \qquad \text{a.e. in } \mathbb{R}^N; \tag{5.13}$$

the constants of equivalence between C_3, C_4 do not depend on ν.

Following the ideas of [39, Theorem 3.4] we prove a convergence Lemma:

Lemma 5.10 *Let (z_n) be a sequence of nonnegative functions, converging a.e. in $L^1(\Omega)$. Extending z_n by 0 in $\mathbb{R}^N \backslash \Omega$, assume that for some $C > 0$,*

$$\int_\Omega z_n^{\frac{q}{p-1}} \xi^{q_*} dx \leqq C \int_\Omega |\nabla \xi|^{q_*} \, dx \qquad \forall n \in \mathbb{N}, \forall \xi \in \mathcal{D}^+(\mathbb{R}^N).$$

Then (z_n) converges strongly in $L^{q/(p-1)}(\Omega)$.

Proof. From our assumption, (z_n) is bounded in $L^{q/(p-1)}(\Omega)$, then up to a subsequence, it converges to some z weakly in $L^{q/(p-1)}(\Omega)$ and *a.e.* in Ω. Consider a ball $B \supset \Omega$ of radius $2 diam\Omega$, and denote by G the Green function associated to $-\Delta$ in B. Set $w_n = z_n^{q/(p-1)}$, and extend w_n by 0 to $\mathbb{R}^N \backslash \Omega$. Then for any compact $K \subset \mathbb{R}^N$,

$$\int_{K \cap \Omega} w_n dx = \int_{K \cap B} w_n dx \leqq C Cap_{1,q*}(K, \mathbb{R}^N),$$

which means that $[w_n]_{1,q^*,B}$ is bounded, and

$$|\nabla G(w_n)(x)| \leqq \int_B |\nabla_x G(x,y)| \, w_n(y) dy \leqq C G_1 * w_n(x),$$

with $C = C(N,\text{diam}\Omega)$. In turn from [39, Corollary 2.5], we get the upperestimate

$$\left[|\nabla G(w_n)|^{\frac{q}{p-1}} \right]_{1,q^*,B} \leqq C \left[|G_1 * w_n|^{\frac{q}{p-1}} \right]_{1,q^*,B} \leqq C \left[w_n \right]_{1,q^*,B}^{q/(p-1)}.$$

Therefore $(|\nabla G(w_n)|)$ is bounded in $L^{q/(p-1)}(B)$, thus $(|\nabla G(w_n - w)|)$ is bounded in $L^{q/(p-1)}(B)$. Let $\varphi \in \mathcal{D}(B)$ and $\varepsilon > 0$ be fixed. Since (z_n) converges *a.e.* to z , from the Egoroff theorem, there exists a measurable set $\omega_\varepsilon \subset B$ such that (w_n) converges to $w = z^{q/(p-1)}$ uniformly on ω_ε, and $\| |\nabla \varphi| \|_{L^{q^*}(B \backslash \omega_\varepsilon)} \leqq \varepsilon$. There holds

$$\begin{aligned} \left| \int_\Omega (w_n - w) \varphi dx \right| &= \left| \int_B (w_n - w) \varphi dx \right| = \\ &= \left| - \int_B (\Delta (G(w_n - w) \varphi dx \right| = \left| - \int_B \nabla (G(w_n - w) . \nabla \varphi dx \right| \end{aligned}$$

Considering the two integrals on $B \backslash \omega_\varepsilon$ and ω_ε we find $\lim \int_\Omega (w_n - w) \varphi dx = 0$. Taking $\varphi = 1$ on Ω, it follows that $\lim \int_\Omega z_n^{q/(p-1)} dx = \int_\Omega z^{q/(p-1)} dx$ and the proof is done. ■

Proof of Theorem 5.8. From our assumption, $\mu \in \mathcal{M}^{q^*}(\Omega)$. We consider the problem associated to $\mu_n = \mu * \rho_n$

$$-\Delta_p u_n + |\nabla u_n|^q = \mu_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega. \tag{5.14}$$

For $q \leqq p$, from [12, Theorem 2.1], as in the proof of Theorem 3.3, (5.14) admits a nonnegative solution $u_n \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Moreover we can approximate u_n in $C^{1,\alpha}(\overline{\Omega})$ by the solution $u_{n,\varepsilon}$ ($\varepsilon > 0$) of the problem

$$-div((\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon}) + (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} = \mu_n \quad \text{in } \Omega, \qquad u_{n,\varepsilon} = 0 \quad \text{on } \partial\Omega.$$

Multiplying this equation by ξ^{q_*} with $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, we obtain

$$\begin{aligned} & q_* \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon}.\xi^{q_*-1}\nabla\xi dx + \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} \xi^{q_*} dx \\ = & \int_\Omega \xi^{q_*}\mu_n dx + q_* \int_{\partial\Omega} \xi^{q_*}(\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon}.\nu ds. \end{aligned}$$

The boundary term is nonpositive, hence going to the limit as $\varepsilon \to 0$, we get

$$\int_\Omega |\nabla u_n|^q \, \xi^{q_*} dx \leqq \int_\Omega \xi^{q_*}\mu_n dx + q_* \int_\Omega |\nabla u_n|^{p-2} \nabla u_n.\xi^{q_*-1}\nabla\xi dx \tag{5.15}$$

When $p = 2$, existence also holds for $q > 2$, from [32]; and then $u_n \in C^2\left(\overline{\Omega}\right)$, thus (5.15) is still true. As in Lemma 4.2, it follows that for any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$

$$\int_\Omega |\nabla u_n|^q \, \xi^{q_*} dx \leqq C(\int_\Omega \xi^{q_*} d\mu_n + \int_\Omega |\nabla\xi|^{q_*} \, dx). \tag{5.16}$$

Otherwise, since $\mu_n(\Omega) \leqq \mu(\Omega)$, from Lemma 2.3, up to a subsequence (u_n) converges *a.e.* to a function u, $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$ and (∇u_n) converges *a.e.* to ∇u in Ω. Note also that (μ_n) is a sequence of good approximations of μ, since μ has a compact support (see [8]). From (4.5), for any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, we have $\lim \int_\Omega \xi^{q_*} d\mu_n = \int_\Omega \xi^{q_*} d\mu$, since $\xi^{q_*} \in C_c(\mathbb{R}^N)$. Then $\int_\Omega \xi^{q_*} d\mu \leqq C \int_\Omega |\nabla\xi|^{q_*} \, dx$. From the Fatou Lemma, we obtain

$$\int_\Omega |\nabla u|^q \, \xi^{q_*} dx \leqq C(\int_\Omega \xi^{q_*} d\mu + \int_\Omega |\nabla\xi|^{q_*} \, dx) \leqq C \int_\Omega |\nabla\xi|^{q_*} \, dx, \tag{5.17}$$

hence $|\nabla u|^q \in L^1(\Omega)$. And then for any compact $K \subset \mathbb{R}^N$, taking $\xi = 1$ on K,

$$\int_{K\cap\Omega} |\nabla u|^q \, dx \leqq C Cap_{1,q_*}(K, \mathbb{R}^N),$$

thus $[|\nabla u|^q]_{1,q^*,\Omega}$ is finite. Moreover, extending μ by 0 to $\mathbb{R}^N \backslash \Omega$, we see from Remark 5.9 that μ satisfies condition (5.11), which is equivalent to (5.13). By convexity, μ_n also satisfies (5.13) and hence (5.11), with the same constants, *i.e.* for any $n \in \mathbb{N}$ and any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$,

$$\int_\Omega \xi^{q_*} d\mu_n \leqq C_2 \int_\Omega |\nabla\xi|^{q_*} \, dx \tag{5.18}$$

Then from (5.16) with another $C > 0$,

$$\int_\Omega |\nabla u_n|^q \, \xi^{q_*} dx \leqq C \int_\Omega |\nabla\xi|^{q_*} \, dx \tag{5.19}$$

Next we can apply Lemma 5.10 to $z_n = |\nabla u_n|^{p-1}$, since (∇u_n) converges *a.e.* to ∇u in Ω. Then $(|\nabla u_n|^q)$ converges strongly in $L^1(\Omega)$ to $|\nabla u|^q$. Thus $(\mu_n - |\nabla u_n|^q)$ is a good approximation of $(\mu - |\nabla u|^q)$. From Theorem 2.6, u is a R-solution of the problem.

From [25, Theorem 1.4], condition (5.17) (for $N \geqq 2$) implies that $q_* < N$, that means $q > q_c$, or $|\nabla u|^q = 0$ in Ω, thus $\mu = 0$. If $\mu = divg$ with $g \in (L^{N(q+1-p)/(p-1),\infty}(\Omega))^N$ with compact support, then $|g|^{\frac{q}{p-1}} \in L^{N/q_*,\infty}(\Omega)$, thus

$$\int_\Omega \zeta^{q_*} \, |g|^{\frac{q}{p-1}} \, dx \leq C_2 \int_\Omega |\nabla\zeta|^{q_*} \, dx, \qquad \forall\zeta \in \mathcal{D}^+(\Omega).$$

Hence μ satisfies (5.11) from the Hölder inequality. Note that $\mu \in \mathcal{M}^{q_*}(\Omega)$, since $q > q_c$ implies $|g| \in L^{q/(p-1)}(\Omega)^N$. ∎

Remark 5.11 *Let $q \leqq p$ and $\mu = divg$, where g has a compact support in Ω. From Theorems 5.5 and 5.8, we have existence when $g \in (L^{p'}(\Omega))^N$, or when $g \in (L^{N(q+1-p)/(p-1),\infty}(\Omega))^N$. Observe that $L^{p'}(\Omega) \supset L^{N(q+1-p)/(p-1)}(\Omega)$ if and only if $\tilde{q} \leqq q$, where $\tilde{q}$ is defined at (1.6). Hence Theorem 5.5 brings better results than Theorem 5.8 when $\tilde{q} \leqq q \leqq p$.*

Remark 5.12 *The extension of this result to the case $p < q, p \neq 2$ will be studied in a further article.*

6 Some regularity results

In this section we give some regularity properties for the problem:

$$-\Delta_p u + H(x, u, \nabla u) = 0 \qquad \text{in } \Omega. \tag{6.1}$$

We first recall some local estimates of the gradient for renormalized solutions, see [19], following the first results of [9], and many others, see among them [3], [26].

Lemma 6.1 *Let u be the R-solution of problem*

$$-\Delta_p u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $f \in L^m(\Omega)$, $1 < m < N$. Set $\overline{m} = Np/(Np - N + p) = p/\tilde{q}$, where $\tilde{q}$ is defined in (1.6).

(i) If $m > N/p$, then $u \in L^\infty(\Omega)$. If $m = N/p$, then $u \in L^k(\Omega)$ for any $k \geqq 1$. If $m < N/p$, then $u^{p-1} \in L^k(\Omega)$ for $k = Nm/(N - pm)$.

(ii) $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$, where $m^* = Nm/(N - m)$. If $\overline{m} \leqq m$, then $u \in W_0^{1,p}(\Omega)$.

Remark 6.2 *The estimates on u and $|\nabla u|$ are obtained in the case $m < \overline{m}$ by using the classical test functions $\phi_{\beta,\varepsilon}(T_k(u))$, where $\phi_{\beta,\varepsilon}(w) = \int_0^w (\varepsilon + |t|)^{-\beta} dt$, for given real $\beta < 1$. Let us recall the proof in the case $m \geqq \overline{m}$, $p < N$. Then $L^m(\Omega) \subset W^{-1,p'}(\Omega)$, thus, from uniqueness, $u \in W_0^{1,p}(\Omega)$ and u is a variational solution. If $m = \overline{m}$, then $m^* = p'$, and the conclusion follows. Suppose $m > \overline{m}$, equivalently $m^* > p'$. For any $\sigma > p$, for any $F \in (L^\sigma(\Omega))^N$, there exists a unique weak solution w in $W_0^{1,\sigma}(\Omega)$ of the problem*

$$-\Delta_p w = div(|F|^{p-2} F) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

see [22], [29], [30]. Let v be the unique solution in $W_0^{1,1}(\Omega)$ of the problem

$$-\Delta v = f \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

Then from the classical Calderon-Zygmund theory, $v \in W^{2,m}(\Omega)$, then $|\nabla v| \in L^{m^*}(\Omega)$. Let F be defined by $|F|^{p-2} F = \nabla v$. Then $F \in (L^\sigma(\Omega))^N$, with $\sigma = (p-1)m^* > p$. Then $-\Delta_p w = -\Delta v = f$, thus $w = u$. Then $u \in W_0^{1,\sigma}(\Omega)$, thus $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$.

We also obtain local estimates:

Lemma 6.3 *Let* $u \in W^{1,p}_{loc}(\Omega)$ *such that*

$$-\Delta_p u = f \quad in\ \Omega,$$

with $f \in L^m_{loc}(\Omega)$, $1 < m < N$, *and* $m > \overline{m}$. *Then* $|\nabla u|^{p-1} \in L^{m^*}_{loc}(\Omega)$. *Furthermore, for any balls* $B_1 \subset\subset B_2 \subset\subset \Omega$, $\left\| |\nabla u|^{p-1} \right\|_{L^{m^*}(\overline{B_1})}$ *is bounded by a constant which depends only on* N, p, B_1, B_2 *and* $\|u\|_{W^{1,p}(B_2)}$.

Proof. We consider again the function v defined in (6.2), and set $|F|^{p-2} F = \nabla v$. Then $F \in (L^\sigma(\Omega))^N$ with $\sigma = (p-1)m^*$, and $u \in W^{1,p}_{loc}(\Omega)$ is a solution of the problem

$$-\Delta_p u = div(|F|^{p-2} F) \quad \text{in } \Omega.$$

Then, from [29], $u \in W^{1,\sigma}_{loc}(\Omega)$ and for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{W^{1,\sigma}(B_1)}$ is controlled by the norm $\|u\|_{W^{1,p}(B_2)}$. ■

Next we consider problem (6.1) in the case $q < \tilde{q}$, where $\tilde{q}$ is defined at (1.6).

Theorem 6.4 *Let* $0 < q < \tilde{q}$, $N \geqq 2$. *Let* H *be a Caratheodory function on* $\Omega \times \mathbb{R}$ *such that*

$$|H(x, u, \xi)| \leqq g(x) + C\,|\xi|^q\,, \tag{6.3}$$

where $g \in L^{N+\varepsilon}_{loc}(\Omega)$, $C > 0$. *Let* $u \in W^{1,p}_{loc}(\Omega)$ *be any weak solution of problem (6.1). Then* $u \in C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$. *Moreover for any balls* $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{C^{1,\alpha}(\overline{B_1})}$ *is bounded by a constant which depends only on* N, p, B_1, B_2, $\|g\|_{L^{N+\varepsilon}(B_2)}$, *and the norm* $\|u\|_{W^{1,p}(B_2)}$.

Proof. Since $u \in W^{1,p}_{loc}(\Omega)$, the function $f = -H(x, u, \nabla u)$ satisfies $f \in L^{m_0}_{loc}(\Omega)$ from (6.3), with $m_0 = p/q > 1$. Notice that $q < \tilde{q}$ is equivalent to $m_0 > \overline{m}$. If $m_0 > N$, then from [18, Theorem 1.2], $|\nabla u| \in L^\infty_{loc}(\Omega)$ and we get an estimate of $\||\nabla u|\|_{L^\infty(B_1)}$ in terms of the norm $\|u\|_{W^{1,p}(B_2)}$ and $\|g\|_{L^{N+\varepsilon}(B_2)}$. Then $u \in C(\Omega)$, $f \in L^\infty_{loc}(\Omega)$, hence $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, see [43].

Next suppose that $m_0 < N$. Then from Lemma 6.3 we have $|\nabla u|^{p-1} \in L^{(p-1)m_0^*}(\Omega)$. In turn, from (6.3), $f \in L^{m_1}_{loc}(\Omega)$ with $m_1 = (p-1)m_0^*/q$. Note that $m_1/m_0 = N(p-1)/(qN-p) > 1$ since $q < \tilde{q}$. By induction, starting from m_1, as long as $m_n < N$, we can define $m_{n+1} = (p-1)m_{n-1}^*/q$, and we find $m_n < m_{n+1}$. If $m_n < N$ for any n, then the sequence converges to $\lambda = N(q-p+1)/q$, which is impossible since $p/q < \lambda$ and $q < \tilde{q}$. Then there exists n_0 such that $m_{n_0} \geqq N$. If $n_0 = N$, or if $m_0 = N$ we modify a little m_0 in order to avoid the case. Then we conclude from above. ■

Remark 6.5 *The result, which holds without any assumption on the sign of* H, *is sharp. Indeed for* $\tilde{q} < q < p < N$, *the problem* $-\Delta_p u = |\nabla u|^q$ *in* $B(0,1)$ *with* $u = 0$ *on* $\partial B(0,1)$ *admits the solution*

$$x \longmapsto u_C(x) = C(|x|^{-\frac{p-q}{q+1-p}} - 1),$$

for suitable $C > 0$, *and* $u_C \in W^{1,p}_0(\Omega)$ *for* $\tilde{q} < q$.

Next we consider the absorption case, and for simplicity the model problem:

Theorem 6.6 *Let $p-1<q$. Let u be a nonnegative LR solution of*

$$-\Delta_p u + |\nabla u|^q = 0 \quad in\ \Omega.$$

Then $u \in L^\infty_{loc}(\Omega) \cap W^{1,p}_{loc}(\Omega)$, and for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{L^\infty(B_1)}$ and $\|u\|_{W^{1,p}(B_1)}$ are controlled by the norm $\|u\|_{L^\ell(B_2)}$ for any $\ell \in (p-1, Q_c)$. As a consequence if $q \leqq p$, then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. In particular $\||\nabla u|\|_{L^\infty(B_1)}$ is controlled by $\|u\|_{L^l(B_2)}$.

Proof. Since $-\Delta_p u \leqq 0$ in Ω, then $u \in L^\infty_{loc}(\Omega)$ from [28], and u satisfies a weak Harnack inequality: for almost any x_0 such that $B(x_0, 3\rho) \subset \Omega$, and any $\ell \in (p-1, Q_c)$,

$$\sup_{B(x_0,\rho)} u \leq C \left(\fint_{B(x_0,2\rho)} u^\ell \right)^{\frac{1}{\ell}}, \tag{6.4}$$

with $C = C(N,p,\ell)$. Then in $B(x_0,\rho)$, $u = T_k(u)$ for some $k>0$, thus $u \in W^{1,p}_{loc}(\Omega)$. For any $\xi \in \mathcal{D}(\Omega)$, taking $u\xi^p$ as a test function, we get

$$\begin{aligned} \int_\Omega |\nabla u|^p \xi^p dx + \int_\Omega |\nabla u|^q u\xi^p dx &= -p \int_\Omega \xi^{p-1} u |\nabla u|^{p-2} \nabla u.\nabla\xi dx \\ &\leqq \frac{1}{2}\int_\Omega |\nabla u|^p \xi^p dx + C_p \int_\Omega u^p |\nabla \xi|^p dx. \end{aligned}$$

Then for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, we obtain that $\||\nabla u|\|_{L^p(B_1)}$ is bounded by a constant which depends only on N, p, B and $\|u\|_{L^\ell(B_2)}$. If $q \leqq p$, we deduce that $u \in C^{1,\alpha}(\Omega)$ and estimates of $|\nabla u| \in L^\infty_{loc}(\Omega)$ from the classical results of [43]. ■

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